WRITTEN REPORT IN LEARNING GEOMETRY: EXPLANATION AND ARGUMENTATION

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In this article, we examine how the written report, within the context of assessment for learning, helps students in learning geometry and in developing their explanation and argumentation skills. We present the results of a qualitative case study involving Portuguese 8th graders. This study suggests that using written reports improves those capabilities and, therefore, the comprehension of geometric concepts and processes. These benefits for learning are enhanced through the implementation of some assessment strategies, namely oral and written feedback.

Key-words: Geometric thinking, explanation, argumentation, assessment for learning, written reports.

INTRODUCTION

Explanation, argumentation and proof are mathematics activities that assume a main role in the teaching and learning of geometry, but present a lot of difficulties to students (Battista, 2007). The need to implement an assessment that contributes to students' learning is also widely recognized: an assessment that guides the students and helps them to improve their learning (Wiliam, 2007). As such, in this study, we attempted to understand how the written report, as a tool of assessment for learning, contributes to learning geometry and, in particular, reinforces the development of students' explanation and argumentation processes.

The present study follows from a wider one that aimed at understanding the key role of the written report as an assessment tool supporting the learning of 8th grade students (aged thirteen) in mathematics. The larger study was developed during the academic year 2007/2008 under the scope of project AREA [1].

EXPLANATION, ARGUMENTATION AND PROOF IN TEACHING AND LEARNING GEOMETRY

All over the world and in Portugal, in particular, the mathematics curriculum recognizes geometry as a privileged field for the development of explanation, argumentation and proof (NCTM, 2000; DGIDC, 2007). Battista and Clements (1995) notice the need to shape the curriculum in order to develop students students' explanation and argumentation skills and so that students use proof to justify powerful ideas. According to Polya (1957) mathematical proof should be taught because it helps in: (i) acquiring the notion of intuitive proof and logical reasoning; (ii) understanding a logical system; and (iii) keeping what is learnt in one's memory.

Many authors have addressed geometrical thought based on Van Hiele's model. This model proposes a sequential progression in learning geometry through five discrete and qualitatively different levels of geometrical thinking: visual, descriptive/analytic, abstract/relational, formal deduction and rigor. However, according to Freudenthal (1991), these are relative levels, not absolute ones. Nevertheless, "the levels can help to find and further develop appropriate tasks (…) and they are obviously helpful for explorative activities to come across new, maybe even innovative ideas" (Dorier *et al*., 2003, p. 2). This progression is determined by the teaching process, thus the teacher has a key role in setting appropriate tasks so that students may progress to higher levels of thought and walk towards proof. The learning of deductive proof in mathematics is complex and its progress is neither linear nor free of difficulties (Küchemann & Hoyle, 2002, 2003). Concerning explanation, we may consider several modes, including non-explanations (where, for example, students refer to the teacher's authority), explaining how, explaining to someone else (spontaneously) and explaining to oneself (in response to a question) (Reid, 1999). Argumentation is view as an intentional explication of the reasonings used during the development of a mathematical tasks (Forman *et al*., 1998).

ASSESSMENT FOR LEARNING

Current mathematics curriculum documents advocate an assessment whose main purpose is to support students' learning, and whose forms constitute, at the same time, learning situations (DGIDC, 2007; NCTM, 1995, 2000). "Assessment in education must, first and foremost, serve the purpose of supporting learning" (Black & Wiliam, 2006, p. 9). In this study, assessment for learning is seen as "all the intent that, acting on the mechanisms of learning, directly contributes to the progression and/or redirection of learning" (Santos, 2002, p. 77). Several studies show that the focus on assessment for learning, as opposed to an assessment of learning, may produce substantial improvement in the performance of students (Black & William, 1998).

In order to develop their own knowledge about thinking mathematically, students need to develop a conscious, reflective practice, which encompasses the processes of self-assessment. According to Hadji (1997), self-assessment is an activity of reflected self-control over actions and behaviours on behalf of the individual who is learning. Santos (2002) stresses that self-assessment implies that one becomes aware of the different moments and aspects of his/her cognitive activity, therefore it is a meta-cognitive process. A non-conscious self-control action is a tacit, spontaneous activity that is natural in the activity of any individual (Nunziati, 1990), and in this sense all human beings self-assess themselves. Meta-cognition goes beyond non-conscious self-control, for it is a conscious and reflective action (Nunziati, 1990).

Some assessment strategies can be adopted to promote learning, including: a positive approach of the error; oral questioning of students; feedback; negotiation of assessment criteria; and the use of alternative and diversified assessment instruments (Black *et al*., 2003; Santos, 2002). In particular, the written report is a privileged instrument to monitor students' learning. Students' work on written reports has advantages in terms of developing their explanation and argumentation skills, which are two intrinsic requests of this instrument; furthermore, written reports may help students to reflect upon their work, because time and space are given (Mason, Burton & Stacey, 1982). “Intensive approach to argumentative skills, relevant for mathematical argumentation, seems to be possible through an interactive management of students’ approach to writing” (Douek & Pichat, 2003). The description of thinking processes, with the identification of the strategies used to solve a given task, including the difficulties that were encountered and the mistakes that were made, allows students to rethink their learning process. However, it is desirable that a report be done in “two stages” to allow for an effective opportunity for learning. This means that a first version of the report is subject to the teacher’s feedback and then the student develops a new version, a second one, taking into account the feedback received (Pinto & Santos, 2006).

METHODOLOGY

This study was based on an interpretative paradigm and on a qualitative approach. We chose the case study for the design research, given the nature of the problem to study and the desired end product (Yin, 2002).

The research involved an 8th grade class, with 24 students. We selected four of these students based on different mathematical performances, and taking into account their mathematics communication skills. These students were Maria, Rute, Duarte, and Telmo, and they constituted a working group in the classroom.

Data were collected through lesson observation, namely, the lesson dedicated to the discussion of the guidelines for preparing the report and of the assessment criteria, and the lessons dedicated to carrying out tasks as well as the first and second versions of the reports. Three individual interviews to each of the four students were made, the first one at the beginning of the school year and the others after the establishment of the second version of each report. Two tasks led to the development of two written reports, each one with two versions.

The data were subjected to several levels of analysis that took place periodically (Miles & Huberman, 1994), based on categories defined a posteriori that arose from the data gathered, keeping in mind the focus of the study and the theoretical framework.

PEDAGOGICAL CONTEXT

Since the writing of a report was a novelty for the students, they were given a set of guidelines for writing the report and the assessment criteria. These two documents were discussed with the students. According to the guidelines, the organization of the report should include three parts: introduction, development, and conclusion. The first two parts should be produced within the group, as the tasks that give rise to the reports. The last part should be held individually and it included students’ self-assessment. The reports were produced in two "stages", the students benefiting from the teacher's comments to the first stage in order to improve the second one. Students were not required to do any proof, but rather to provide explanations for their thinking (Küchemann & Hoyle, 2003).

The first task proposed an investigation of possible generalizations of the Pythagorean theorem. Students were asked to remember and to reflect upon the relationship between the areas of the squares constructed on the sides of a right triangle, and to investigate what happens if they construct other geometric figures on the sides of a right triangle. The second task was a problem that involves the application of the Pythagorean theorem in space. Students were asked to construct a cone based on one of the three equal sectors of a circle, with a radius of six centimetres, and to determine the height of the constructed cone. They were also encouraged to explain how they could determine the height of a cone obtained from a circle with a radius r. These tasks were chosen based on the assumption that presenting students with unfamiliar questions can provide a rich context for classroom discussion which helps students in developing mathematical arguments (Küchemann & Hoyle, 2003).

The first report

In the first task, students reflect on the meaning and implications of the Pythagorean theorem and review some geometric concepts and procedures (such as what an equilateral triangle is and how it can be constructed with ruler and compass). Due to the nature of the task, the group is still required to formulate and test conjectures, and to argue in favour of their ideas, thus appealing to students' mathematical reasoning skills. In particular, when writing the report, the students, in group, explain how they exploited the first situation proposed in the task, concerning equilateral triangles built on the sides of a right triangle.

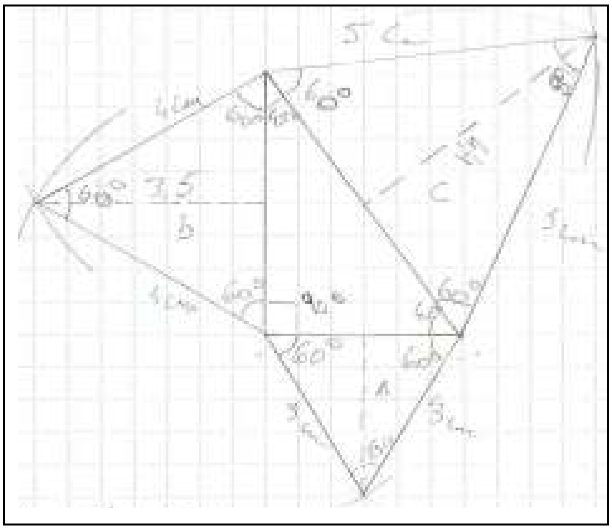
In the first version of their report, students described how they had built the equilateral triangles and stated how they had determined the areas of those triangles:

We started by making a right triangle, with the help of a compass we drew around it (at the endpoints of the right triangle) three equilateral triangles, because we couldn't obtain equilateral triangles using with a rules nor a good graphic design. We determined the area of the triangles.

The justification for the use of compass comes in the wake of some oral feedback provided during the preparation of the report. This feedback may have helped the students to explain their options:

Rute:	We did it like this: with the help of the compass, we made around it three equilateral triangles. Then we can put… ah…
Teacher:	Why did you use the compass?
Rute:	Because we couldn't complete the task with the ruler only.
Teacher:	So, couldn't you draw a triangle with the ruler only?

Rute: Yes, but in order to be an equilateral triangle, it had to have all equal sides.



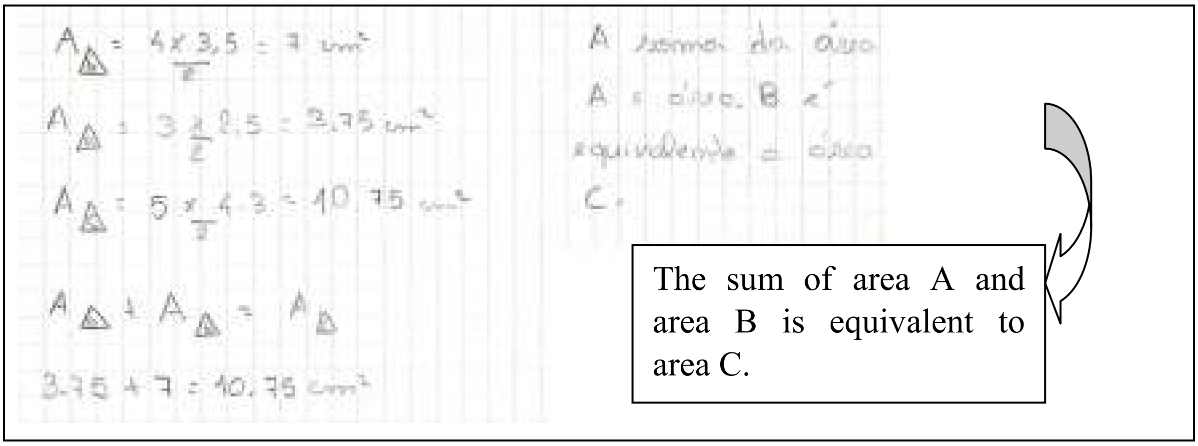
In an attached document to their report, the group presented the construction of equilateral triangles, as well as the values of the basis and the height considered in each one. It also presented the calculations that were made to determine the corresponding areas.

However, in any part of the report, did the students explain how they had found the values of the bases and heights, nor what conclusions they obtained from the areas determined. Two different comments were provided to the first version of the report. On one hand, the teacher praised students for their use of a compass and the reasons for their choice: "You did an excellent option. It’s a good way to answer a problem that you had to overcome." In this way, the teacher identified positive aspects of the report, so that knowledge could be consciously recognized by students and their self confidence could be promoted (Santos, 2003). On the other hand, the teacher questioned students about the conclusions they had drawn from the areas obtained: "And what did you find?". Furthermore, the teacher still posed some questions written near the construction of the triangles, which sought to guide the work of students in order to include the missing information in the report: "How did you come to these figures? Which relationship may you establish?"

While working on the second version of their report, the students kept the description that had been praised and tried to answer the questions. They explained in more detail how they had proceeded, namely in finding the values of the basis and height of the triangles, in determining the corresponding areas in each equilateral triangle, and in making explicit the conclusions they had obtained for the first situation:

> We determined the area of the triangles. We know that in order to determine the area of a triangle: $\frac{basis \times height}{2}$, we measure the height and the basis, we multiplied and then we divided by 2 (and likewise for the three triangles). We concluded that the sum of area A and area B is equal to area C.

In the final version, the students determined and identified the value of the area of each one of the considered triangles and explained the relationship found among the areas of the equilateral triangles constructed on the sides of the right triangle. This work was based on the figure of the first version:



Students still added a comment. They identified the negative aspects of the first version and they improved them in the second stage: “[In the first stage] we didn’t present the value for the areas, we messed up the computations, and we did not present the conclusions.” The students identified and corrected their own mistakes.

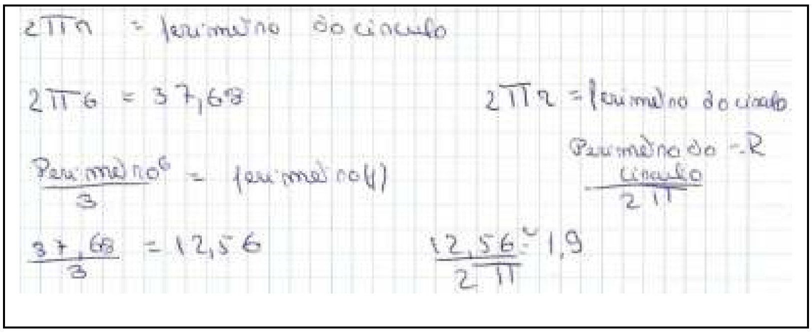
The second report

In the second task, the students review and apply the Pythagorean theorem as well as some mathematical concepts and procedures (such as, the height of a cone or the perimeter of a circle given its radius). Due to the nature of the task, it calls, mostly, for problem-solving and mathematical reasoning skills.

In the report, the students explained how they had built the cones and sought reasons for their actions. In particular, they explain how to determine the angle of each of the three circular sectors:

> We started by reading the task and answering to what had been requested. We drew a circle of radius 6 cm. To divide the angle into three equal parts, we know that the angle measures 360º: (so $\frac{360^{\circ}}{3} = 120^{\circ}$). With the help of a protractor, we measured, on the radius, 120º three times and joined the points and we got 3 equal parts. Then, we cut the three parts, and with the help of some tape, we constructed three cones.

Next, the students described the strategy implemented to determine the height of the cones. Before moving to the resolution itself, they made a brief description of how the group had addressed the issue, referring various ideas discussed and some difficulties encountered, which they sought to overcome with the help of the teacher. Then they determined the radius of the basis of the cone, giving the necessary calculations (determining the perimeter of the original circle, the perimeter of the basis of the cone and, finally, the radius of the basis of the cone).



However they did not explain the calculations nor did they give reasons for those calculations; they did not distinguish the two circles involved (the original one and the basis of the cone), nor did they present units of measurement. Written feedback was provided with the intention of alerting students to these aspects: "Why did you do these calculations? You refer the perimeter of the circle several times. Maybe it wold be better to distinguish which circle you are talking about in each situation. Attention to the lack of measurement units". The importance of students' explanation and justification of their calculations was further strengthened through oral feedback:

Teacher:	"(...) you must try to explain the calculations you presented better and why you have done them". You presented these calculations, didn't you? For what? When? How?
Rute:	The teacher wants to know everything!
Teacher:	I want to know everything, no… Imagine that I'm teaching a lesson and I write something on the blackboard, and then you ask me "teacher, what is that?" and I say "You want to know everything!", right?
Rute:	Teacher, but, here, we already know that this is the perimeter...
Teacher:	You know, but you must write what you mean. I am not going to take Rute home to explain to me, right?

It was also necessary to complement the written feedback with new clues, so that the students could distinguish the different circles considerered in the resolution of the problem:

Rute:	Teacher, how do we distinguish the circles?
Teacher:	Which circles did you work with?
Rute:	With the one with radius six.
Teacher:	Yes. And didn't you work with any other circle?
Rute:	With the basis.
Teacher:	The basis?
Rute:	Yes, of the cone.
Teacher:	So, in the report, you only have to say which one you are referring to when you explain what you did.

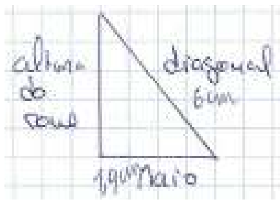
The students took into account the feedback received, both oral and written. In the final version of the report, besides adding the measurement units, they described how they had proceeded to determine the radius of the basis of the cone. They clarified the context, they explained the purpose of the calculations they had presented, and they also identified the circle referred in each case:

> First we found the perimeter of the circle of the problem. Then we divided the perimeter of the circle of the problem into three equal parts, and we got the perimeter of the basis of

a cone. Knowing that to find the perimeter of the circle is $2\pi r$, to find the radius is the other way around: $P \div 2\pi = r$. And then, we obtained 1,9 cm.

In the first version of the report, students had already tried to describe in detail the right triangle used to determine the height of the cone and they explained how they had determined the length of the hypotenuse (which they refer to as diagonal) of that triangle:

If we draw the height of the cone, it will coincide with the radius forming an angle of 90º. If, at the endpoints of the lines, we draw a line segment, it will form a right triangle and, for our own luck, it was the diagonal, which we knew about.



We know that the diagonal measures 6 cm because the diagonal is the radius of the circle when we open the cone, and, as the radius of the circle is 6 cm, we got to know the diagonal.

Finally, the students presented the necessary calculations to determine the height of the cone, but they did not mention how they had concluded that “height of the cone^2 = diagonal^2 - radius^2”. They were reminded of this fact through written feedback: "How do you achieve this equality?" In the final version of the report, the students considered the feedback received and stated that they had used the Pythagorean theorem to obtain the height of the cone.

DISCUSSION OF RESULTS

In this study, students were asked to describe and explain the strategies used in the implementation of two tasks and to submit the results, duly substantiated, under the form of written reports. Students, working in a group, were given constructive comments on the first version of their reports so that they could improve their work and develop a second version. In many cases, in the first version of the reports, students gave procedural explanations instead of providing a mathematical justification (Hoyle & Küchemann, 2003). In other words, they presented how they had done their work, but not why. For example, in the first version of the report regarding the first task, students described how they had built the equilateral triangle, but they did not mention the characteristics of this figure. In the second version of the report, students presented mathematical arguments for the choices made and for the results found in performing the tasks. They also used symbolic language of mathematics when necessary (it happened, for example, when they obtained the area of equilateral triangles in the first task or when they obtained the height of the cone in the second task). However, in both cases, they seemed to be, mainly, at the descriptive/analytic level of Van Hiele’s geometrical thinking model.

Feedback, whether oral or written, allowed students to identify aspects to improve in the reports and provided clues about what students could do to develop their first productions. Indeed, feedback seems to have enabled students to produce a better report in the second version, especially regarding explanation and justification of the strategies adopted (it should be noted, for example, the explanation given, in the final version, to the operation performed in the first phase to obtain the radius of the basis of the cone, starting from its perimeter). In addition, the feedback did not contain any information about errors; it only included guiding questions and comments (Black *et al*., 2003; Santos, 2003). This led students to identify mistakes and to correct them (as is evident in the first task, in which the students relate what they had done wrong in the first version). Thus, feedback also promoted the development of students' reflection and self-assessment skills (Nunziati, 1990).

The need for students to explain and justify, in written form, the mathematical procedures and results involved in performing mathematically rich tasks caused a high level of demand and consequently of learning. These situations, which involve knowledge that students possibly know, but which they need to explain and justify, have a strong didactic purpose (Küchemann & Hoyle, 2003). The identified benefits associated with the written reports seem to be enhanced by investing on a type of report in "two stages", in which oral and written feedback gain prominence.

NOTES

1. The project AREA (Monitoring Assessment in Teaching and Learning) is a research project funded by the Foundation for Science and Technology (PTDC/CED/64970/2006). The main objectives of the project are to develop, implement and study practices of assessment that contribute for learning. Further information can be found in http://area.fc.ul.

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